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EINSTEIN METRICS ON CONNECTED SUMS OF $S^2 \times S^3$

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Abstract

In this paper we construct infinitely many families of Einstein metrics on the connected sums of arbitrary number of copies of $S^2 \times S^3$. We realize these 5-manifolds as total spaces of Seifert bundles over Del Pezzo orbifolds. A Kähler–Einstein metric on the Del Pezzo orbifold is then lifted to an Einstein metric using the Kobayashi–Boyer–Galicki method.

It is still very poorly understood which 5-manifolds carry an Einstein metric with positive constant. By Myers' theorem, the fundamental group of such a manifold is finite, therefore it is reasonable to concentrate on the simply connected case. The most familiar examples are connected sums of k copies of $S^2 \times S^3$.

For $k \leq 9$, Einstein metrics on these were constructed by Boyer, Galicki and Nakamaye [**BGN02**, **BGN03b**, **BG03**]. In this paper we extend their result to any k.

Theorem 1. *For every $k \geq 6$, there are infinitely many $(2k-2)$-dimensional families of Einstein metrics on the connected sum of k copies of $S^2 \times S^3$.*

It was known earlier that these spaces carry metrics of positive Ricci curvature; this is a special case of the results of [**SY91**]. Sasakian metrics of positive Ricci curvature on $k\#(S^2 \times S^3)$ are constructed in [**BGN03a**].

The constructions in [**BGN02**, **BGN03b**, **BG03**] exhibit suitable links of singular hypersurfaces $0 \in Y \subset \mathbb{C}^m$ with $\mathbb{C}^*$-action. These links are Seifert bundles over the corresponding weighted projective hypersurfaces $(Y \setminus \{0\})/\mathbb{C}^*$. The Einstein metrics are then constructed from Kähler–Einstein metrics on these weighted projective hypersurfaces.

Here we look at this construction from the other end. Starting with a projective variety X, we study Seifert bundles $Y \to X$. For X smooth, these are described in [**OW75**]. When X is a surface, the topology of Y can be understood well enough. The freedom we gain is that one can

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start with an arbitrary algebraic surface, not just with a hypersurface in a weighted projective space.

The conditions for the method to work are somewhat delicate, but there are probably many more examples obtained similarly.

The natural setting of this approach is to start with an orbifold X. The main ideas are similar, but this requires a somewhat lengthy study of Seifert bundles over orbifolds (see [**KOL05**]).

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1. Seifert bundles over complex manifolds

Seifert bundles over complex manifolds were introduced and studied in [**OW75**]. The quickest approach is to define them first locally by explicit formulas and then to patch the local forms together. (See [**OW75**] for a conceptually better, though equivalent, definition.)

Definition 2. Let $D^n_{\mathbf{z}} \subset \mathbb{C}^n$ be the unit polydisc with coordinates $z_1, \dots, z_n$. Let G_t be either $\mathbb{C}^*$ or the unit circle $S^1 \subset \mathbb{C}$, both with coordinate t. Pick pairwise relatively prime natural numbers $a_1, \dots, a_n$ and for every i pick $0 < b_i \leq a_i$ such that b_i is relatively prime to a_i. Set $a = \prod a_i$.

Let ϵ_i be a primitive a_ith root of unity and consider the $\mathbb{Z}/a$ action on $D^n_{\mathbf{z}}$ given by

$$\phi : (z_1, \dots, z_n) \mapsto (\epsilon_1 z_1, \dots, \epsilon_n z_n),$$

and its lifting to $G_t \times D^n_{\mathbf{z}}$

$$\Phi : (t, z_1, \dots, z_n) \mapsto \left(\left(\prod \epsilon_i^{b_i} \right) t, \epsilon_1 z_1, \dots, \epsilon_n z_n \right).$$

It is easy to see that

$$D^n_{\mathbf{z}} / \langle \phi \rangle \cong D^n_{\mathbf{x}}, \quad \text{where} \quad x_i = z_i^{a_i}.$$

The quotient map $D^n_{\mathbf{z}} \to D^n_{\mathbf{x}}$ ramifies along $(z_i = 0)$ with multiplicity a_i. Furthermore, $\prod \epsilon_i^{b_i}$ is a primitive ath root of unity, which implies that $G_t \times D^n_{\mathbf{z}} / \langle \Phi \rangle$ is a smooth manifold. The second projection of $G_t \times D^n_{\mathbf{z}}$ descends to a map

$$f : G_t \times D^n_{\mathbf{z}} / \langle \Phi \rangle \to D^n_{\mathbf{z}} / \langle \phi \rangle \cong D^n_{\mathbf{x}}.$$

This is called a *standard Seifert G-bundle* over $D^n_{\mathbf{x}}$ with orbit invariants (a_i, b_i) along $(x_i = 0)$. ((α_i, β_i) in the notation of [**OW75**].)

Notice that the bounded and $\mathbb{Z}/a$-invariant holomorphic sections of $\mathbb{C}_t \times D^n_{\mathbf{z}} \to D^n_{\mathbf{z}}$ are of the form

$$t = \left(\prod z_i^{b_i} \right) \cdot h(z_1^{a_1}, \dots, z_n^{a_n}), \quad \text{where } h \text{ is holomorphic.}$$

Thus the bounded holomorphic sections of $\mathbb{C}_t \times D^n_{\mathbf{z}}/\langle\Phi\rangle \to D^n_{\mathbf{x}}$ form a locally free sheaf whose generator can be thought of as $\prod x_i^{b_i/a_i}$.

Definition 3. Let X be a complex manifold and $D_i \subset X$ smooth divisors intersecting transversally. For every i pick natural numbers $0 < b_i \leq a_i$ such that a_i and b_i are relatively prime for every i. Assume that a_i and a_j are relatively prime whenever $D_i \cap D_j \neq \emptyset$.

Every $x \in X$ has a neighborhood $x \in U \subset X$ and a biholomorphism $\tau : U \cong D^n$ such that every $D_i \cap U$ is mapped to a coordinate hyperplane in D^n by τ (or the intersection is empty). This assigns numbers (a_i, b_i) to every coordinate on D^n. (We set $a_j = b_j = 1$ for those coordinates that do not correspond to a D_i.)

A *Seifert G-bundle* over X with *orbit invariants* (a_i, b_i) along D_i is a real manifold L with a differentiable G-action and a differentiable map $f : L \to X$ such that for every neighborhood U as above, $\tau \circ f : f^{-1}(U) \to U \cong D^n$ is fiber preserving equivariantly diffeomorphic to the corresponding standard Seifert model.

For any $P \in X$, the number $\prod_{P \in D_i} a_i$ is called the *multiplicity* of the Seifert fiber over P.

A Seifert S^1-bundle is also called a *Seifert bundle*,

There is a one–to–one correspondence between Seifert S^1-bundles and Seifert $\mathbb{C}^*$-bundles over X.

Definition 4. Analogously, a *holomorphic Seifert $\mathbb{C}^*$-bundle* over X with orbit invariants (a_i, b_i) along D_i is a complex manifold Y with a holomorphic $\mathbb{C}^*$-action and a holomorphic map $f : Y \to X$ such that for every neighborhood U as above, $\tau \circ f : f^{-1}(U) \to U \cong D^n$ is fiber preserving equivariantly biholomorphic to the corresponding standard Seifert model.

From (2) we see that

$$X \supset V \mapsto \{\text{bounded holomorphic sections of } f \text{ over } V \setminus \cup D_i\}$$

defines a locally free sheaf, denoted by B_Y.

5 (Construction of Seifert bundles, [**OW75**, 3.9])**.** Let X be a complex manifold such that $H_1(X, \mathbb{Z}) = 0$. Assume that we are given

1) smooth divisors $D_i \subset X$ intersecting transversally,
2) natural numbers $0 < b_i \leq a_i$ such that
   a) a_i and b_i are relatively prime for every i, and
   b) a_i and a_j are relatively prime whenever $D_i \cap D_j \neq \emptyset$, and
3) a class $B \in H^2(X, \mathbb{Z})$.

There is a unique Seifert $\mathbb{C}^*$-bundle $f : Y \to X$ such that

4) $f : Y \to X$ has orbit invariants (a_i, b_i) along D_i, and
5) f factors as $f : Y \xrightarrow{\pi} M_Y \xrightarrow{q} X$, where
   a) $q : M_Y \to X$ is the unique $\mathbb{C}^*$-bundle with Chern class $aB + \sum b_i \frac{a}{a_i}[D_i]$ where $a = \operatorname{lcm}\{a_i\}$, and

b) $\pi : Y \to M_Y$ is an a-sheeted branched cover, branching along $q^{-1}(D_i)$ with multiplicity a_i.

Every Seifert $\mathbb{C}^*$-bundle $f : Y \to X$ has a unique such representation. This representation defines the *Chern class* of a Seifert bundle

$$c_1(Y/X) := B + \sum \tfrac{b_i}{a_i}[D_i] \in H^2(X, \mathbb{Q}),$$

where, for a divisor $D \subset X$, $[D] \in H^2(X, \mathbb{Z})$ denotes the corresponding cohomology class. This notation extends to $\mathbb{Q}$-linear combinations of divisors by linearity.

If X is projective and $H^2(X, \mathcal{O}_X) = 0$ then every Seifert $\mathbb{C}^*$-bundle has a unique holomorphic Seifert $\mathbb{C}^*$-bundle structure. It satisfies $c_1(B_Y) = B$.

2. The topology of 5-dimensional Seifert bundles

Notation 6. In this section, X denotes a smooth, projective, simply connected algebraic variety and $D_1, \dots, D_n \subset X$ are smooth divisors intersecting transversally. $f : L \to X$ denotes a Seifert bundle with orbit invariants $(a_1, b_1), \dots, (a_n, b_n)$ along $D_1, \dots, D_n$. Set $a = \operatorname{lcm}(a_1, \dots, a_n)$.

The main result, (10) is for surfaces only, but two of the intermediate steps hold in all dimensions.

Proposition 7. *Notation as in* (6). *Assume that*

1) *the* $[D_i]$ *form part of a basis of* $H_2(X, \mathbb{Z})$,
2) $a \cdot c_1(L) \in H^2(X, \mathbb{Z})$ *is not divisible.*

Then $H_1(L, \mathbb{Z}) = 0$.

If, in addition,

3) $\pi_1(X \setminus (D_1 \cup \dots \cup D_n))$ *is abelian,*

then L *is simply connected.*

Proposition 8. *Notation as in* (6). *Assume that* $\dim X = 2$ *and every* D_i *is a rational curve. Then* $H^3(L, \mathbb{Z})$ *is torsion free.*

Proposition 9. *Notation as in* (6). *Assume that*

1) *the* a_i *are odd, and*
2) $w_2(X) \equiv a \cdot c_1(L)$ *modulo* 2.

Then $w_2(L)$, *the second Stiefel–Whitney class of* L (*cf.* [**MS74**, Sec.4]), *is zero.*

Corollary 10. *Notation as in* (6). *Assume that* $\dim X = 2$ *and the conditions of* (7), (8) *and* (9) *are all satisfied. Then* L *is diffeomorphic to* $(k-1)\#(S^2 \times S^3)$ *for* $k = \dim H^2(X, \mathbb{Q})$.

Proof. By [**Sma62**], a simply connected compact 5–manifold L with vanishing second Stiefel–Whitney class is uniquely determined by $H^3(L,\mathbb{Z})$. q.e.d.

The computation of $H_1(L,\mathbb{Z})$ relies on the following.

Proposition 11 ([**OW75**, 4.6])**.** *Let X be a complex manifold such that $H_1(X,\mathbb{Z}) = 0$ and let $D_1,\dots,D_n \subset X$ be smooth divisors intersecting transversally. Let $f : L \to X$ be a Seifert bundle with invariants $(a_1, b_1,\dots,a_n, b_n, B)$. Then $H_1(L,\mathbb{Z})$ is given by generators $k, g_1,\dots,g_n$ and relations*

1) $a_i g_i + b_i k = 0$ *for* $i = 1,\dots,n$, *and*
2) $k(B \cap \eta) - \sum g_i([D_i] \cap \eta) = 0$ *for every* $\eta \in H_2(X,\mathbb{Z})$.

12 (Proof of (7))**.** It is enough to prove that the equations (11.1–2) have only trivial solution modulo p for every p.

If $p|a_i$ then $p \not| b_i$ so $a_i g_i + b_i k = 0$ gives $k = 0$. Since the D_i form part of a basis of $H_2(X,\mathbb{Z})$, for every j there is an η_j such that $[D_i] \cap \eta_j = \delta_{ij}$. This implies that $g_j = 0$ for every j.

If $p \not| a_i$ for every i then $g_i = -(b_i/a_i)k$ makes sense and the second equation, multiplied through by a, becomes

$$k \cdot \Big(aB + \sum b_i \tfrac{a}{a_i}[D_i]\Big) \cap \eta = 0 \quad \text{for every } \eta \in H_2(X,\mathbb{Z}).$$

By (7.2), $aB + \sum b_i \frac{a}{a_i}[D_i] = ac_1(L)$ is not zero modulo p, so for suitable η we get $k = 0$.

As in [**OW75**, p. 153], the assumption (7.3) implies that $\pi_1(L)$ is abelian. Thus L is simply connected once $H_1(L,\mathbb{Z}) = 0$.

13 (Proof of (8))**.** In order to compute the rest of the cohomology of L, we consider the Leray spectral sequence $H^i(X, R^j f_*\mathbb{Z}_L) \Rightarrow H^{i+j}(L,\mathbb{Z})$. First we get some information about the sheaf $R^1 f_*\mathbb{Z}_L$ and then about the groups $H^i(X, R^1 f_*\mathbb{Z}_L)$.

Proposition 14. *Let $f : L \to X$ be a Seifert bundle.*

1) *There is a natural isomorphism $\tau : R^1 f_*\mathbb{Q}_L \cong \mathbb{Q}_X$.*
2) *There is a natural injection $\tau : R^1 f_*\mathbb{Z}_L \hookrightarrow \mathbb{Z}_X$ which is an isomorphism over the smooth locus.*
3) *If $U \subset X$ is connected then*

$$\tau(H^0(U, R^1 f_*\mathbb{Z}_L)) = m(U) \cdot H^0(U,\mathbb{Z}) \cong m(U) \cdot \mathbb{Z},$$

where $m(U)$ is the lcm *of the multiplicities of all fibers over U.*

Proof. Pick $x \in X$ and a contractible neighborhood $x \in V \subset X$. Then $f^{-1}(V)$ retracts to $f^{-1}(x) \sim S^1$ and (together with the orientation) this gives a distinguished generator $\rho \in H^1(f^{-1}(V),\mathbb{Z})$. This in turn determines a cohomology class $\frac{1}{m(x)}\rho \in H^1(f^{-1}(V),\mathbb{Q})$. These normalized cohomology classes are compatible with each other and give a global section of $R^1 f_*\mathbb{Q}_L$. Thus $R^1 f_*\mathbb{Q}_L = \mathbb{Q}_X$ and we also obtain the injection $\tau : R^1 f_*\mathbb{Z}_L \hookrightarrow \mathbb{Z}_X$ as in (2).

If $U \subset X$ is connected, a section $b \in \mathbb{Z} \cong H^0(U, \mathbb{Z}_U)$ is in $\tau(R^1 f_*\mathbb{Z}_L)$ iff $m(x)$ divides b for every $x \in U$. This is exactly (3). q.e.d.

Corollary 15. *Let $f : L \to X$ be a Seifert bundle with orbit invariants (a_i, b_i) along D_i. Then there is an exact sequence*

$$0 \to R^1 f_*\mathbb{Z}_L \xrightarrow{\tau} \mathbb{Z}_X \to \sum_i \mathbb{Z}_{D_i}/a_i \to 0.$$

Proof. Note that a_i and a_j are relatively prime if $D_i \cap D_j \neq \emptyset$. It is now clear that the kernel of $\mathbb{Z}_X \to \sum_i \mathbb{Z}_{D_i}/a_i$ has the same sections as described in (14.3). q.e.d.

The groups $H^i(X, R^1 f_*\mathbb{Z}_L)$ sit in the long exact cohomology sequence of the short exact sequence of (15). The crucial piece is

$$\sum_i H^1(D_i, \mathbb{Z}/a_i) \to H^2(X, R^1 f_*\mathbb{Z}_L) \to H^2(X, \mathbb{Z}).$$

Thus $H^2(X, R^1 f_*\mathbb{Z}_L)$ is torsion free if $H^1(D_i, \mathbb{Z}) = 0$ for every i.

Therefore, the E_2 term of the Leray spectral sequence $H^i(X, R^j f_*\mathbb{Z}_L)$ $\Rightarrow H^{i+j}(L, \mathbb{Z})$ is

$$\begin{matrix} \mathbb{Z} & * & \mathbb{Z}^k & * & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z}^k & 0 & \mathbb{Z}. \end{matrix}$$

The spectral sequence degenerates at E^3 and we have only two nontrivial differentials

$$\delta_0 : E_2^{0,1} \to E_2^{2,0} \quad \text{and} \quad \delta_2 : E_2^{2,1} \to E_2^{4,0}.$$

In any case, $H^3(L, \mathbb{Z}) \cong \ker \delta_2$ and so it is torsion free.

Note also that if $H_1(L, \mathbb{Q}) = 0$ then δ_0 is nonzero, hence $\operatorname{rank} H^3(L, \mathbb{Q})$ $= \operatorname{rank} H^2(L, \mathbb{Q}) = \operatorname{rank} H^2(X, \mathbb{Q}) - 1$.

16 (Proof of (9))**.** Let $Y \supset L$ denote the corresponding Seifert $\mathbb{C}^*$-bundle. L is an orientable hypersurface, hence its normal bundle is trivial. This implies that $w_i(L) = w_i(Y)|_L$. Since $w_2(Y) \equiv c_1(Y)$ mod 2 (cf. [**MS74**, 14-B]), it is enough to prove that $K_Y = -c_1(Y)$ is divisible by 2.

Let $E \to X$ be the unique holomorphic line bundle with $c_1(E) = ac_1(L)$ with zero section $X \subset E$. Let $M := E \setminus$ (zero section) be the corresponding $\mathbb{C}^*$-bundle. By (5.5.b), there is a branched covering $\pi : Y \to M$ with branching multiplicities a_i. These are all odd, hence by the Hurwitz formula, $K_Y \equiv \pi^* K_M \mod 2$.

By the adjunction formula $K_X = K_E|_X + c_1(E)$. Thus, working in $H^2(X, \mathbb{Z}/2)$, we get that

$$K_E|_X = K_X - c_1(E) = K_X - ac_1(L) \equiv w_2(X) - ac_1(L) \equiv 0 \quad \text{modulo } 2,$$

the last equality by (9.2). The injection $X \hookrightarrow E$ is a homotopy equivalence, thus K_E and hence also K_M are both divisible by 2.

3. Einstein metrics on Seifert bundles

A method of Kobayashi [**Kob63**] (see also [**Bes87**, 9.76]) constructs Einstein metrics on circle bundles $M \to X$ from a Kähler–Einstein metric on X. This was generalized to Seifert bundles $f : L \to X$ in [**BG00**], but in this case one needs an orbifold Kähler–Einstein metric on X and a Hermitian metric on Y.

Definition 17. Let $f : Y \to X$ be a Seifert $\mathbb{C}^*$ bundle. A *Hermitian metric* on Y is a C^∞ family of Hermitian metrics on the fibers.

We can also think of this as a degenerate Hermitian metric h on the line bundle B_Y. On $X \setminus \cup D_i$ we get a Hermitian metric. On a local chart described in (2), let $s(x_1, \dots, x_n)$ be a generating section of B_Y. Then we have the requirement

$$h(s,s) = (C^\infty\text{-function}) \cdot \prod (x_i \bar{x}_i)^{b_i/a_i}.$$

The equivalence is clear since by (2) we can think of s as $(C^\infty\text{-function})\cdot \prod x_i^{b_i/a_i}$.

This again leads to the Chern class equality $c_1(B) + \sum \frac{b_i}{a_i}[D_i] = c_1(L)$.

Definition 18. Let X be a complex manifold and $D_i \subset X$ smooth divisors intersecting transversally. For every i pick a natural number a_i. Assume that a_i and a_j are relatively prime whenever $D_i \cap D_j \neq \emptyset$.

As in (2) and (4), for every $P \in X$ choose a neighborhood $P \in U_P \subset X$ biholomorphic to $D^n_{\mathbf{x}}$ which can be written as a quotient

$$\pi_P : D^n_{\mathbf{z}} \to D^n_{\mathbf{z}}/\langle \phi_P \rangle \cong D^n_{\mathbf{x}} \cong U_P,$$

where $\langle \phi_P \rangle$ is the cyclic group of order $a_P = \prod_{P \in D_i} a_i$ and π_P branches exactly along the divisors $D_i \cap U_P$ with multiplicity a_i.

Thus the action is

$$\phi_P : (z_1, \dots, z_n) \mapsto (\epsilon_1 z_1, \dots, \epsilon_n z_n),$$

where ϵ_j is a primitive a_{i_j}th root of unity for $D_{i_j} \cap U_P \neq \emptyset$ and we set $\epsilon_j = 1$ for those coodinates that do not correspond to any D_i.

An *orbifold Hermitian metric* on $(X, \sum (1 - \frac{1}{a_i}) D_i)$ is a Hermitian metric h on $X \setminus \cup D_i$ such that $\pi_P^* h$ extends to a Hermitian metric on $D^n_{\mathbf{z}}$ for every $P \in X$.

On a local chart described in (2) this means a metric $\sum h_{ij} dx_i \otimes d\bar{x}_j$ defined on $D^n_{\mathbf{x}} \setminus (\prod x_i = 0)$ such that $\sum a_i a_j h_{ij} z_i^{(a_i-1)} \bar{z}_j^{(a_j-1)} dz_i \otimes d\bar{z}_j$ is a metric on $D^n_{\mathbf{z}}$. Thus the orbifold canonical class is $K_X + \sum (1 - \frac{1}{a_i}) D_i$.

An *orbifold Kähler–Einstein metric* is now defined as usual.

Theorem 19 ([**Kob63**, **BG00**])**.** *Let $f : L \to X$ be a Seifert bundle with orbit invariants $(a_1, b_1), \dots, (a_n, b_n)$. L admits an S^1-invariant Einstein metric with positive constant if the following hold.*

1) *The orbifold canonical class $K_X + \sum(1 - \frac{1}{a_i})D_i$ is anti ample and there is a Kähler–Einstein metric on $(X, \sum(1 - \frac{1}{a_i})D_i)$.*
2) *The Chern class of L is a negative multiple of $K_X + \sum(1 - \frac{1}{a_i})D_i$.*

The main impediment to apply (19) is the current shortage of existence results for Kähler–Einstein metrics on orbifolds. We use the following sufficient algebro–geometric condition. There is every reason to expect that it is very far from being optimal, but it does provide a large selection of examples.

In this paper we use (20) only for surfaces. The concept *klt* is defined in (21).

Theorem 20 ([**Nad90**, **DK01**])**.** *Let X be an n-dimensional compact complex manifold and $D_i \subset X$ smooth divisors intersecting transversally.*

Assume that $-(K_X + \sum(1 - \frac{1}{a_i})D_i)$ is ample and there is an $\epsilon > 0$ such that

$$\left(X, \tfrac{n+\epsilon}{n+1}F + \sum(1 - \tfrac{1}{a_i})D_i\right) \quad \text{is klt} \tag{20.1}$$

for every positive $\mathbb{Q}$-linear combination of divisors $F = \sum f_i F_i$ such that $[F] = -[K_X + \sum(1 - \frac{1}{a_i})D_i] \in H^2(X, \mathbb{Q})$.

Then there is orbifold Kähler–Einstein metric on $(X, \sum(1 - \frac{1}{a_i})D_i)$.

If $\sum a_i D_i$ is invariant under a compact group G of biholomorphisms of X, then it is sufficient to check (20.1) *for G-equivariant divisors F.*

Definition 21 (cf. [**KM98**, 2.34])**.** Let X be a complex manifold and D an effective $\mathbb{Q}$-divisor on X. Let $g : Y \to X$ be any proper bimeromorphic morphism, Y smooth. Then there is a unique $\mathbb{Q}$-divisor $D_Y = \sum e_i E_i$ on Y such that

$$K_Y + D_Y \equiv g^*(K_X + D) \quad \text{and} \quad g_* D_Y = D.$$

We say that (X, D) is *klt* (short for Kawamata log terminal) if $e_i < 1$ for every g and for every i.

It is quite hard to check using the above definition if a pair (X, D) is klt or not. For surfaces, there are reasonably sharp multiplicity conditions which ensure that a given pair (X, D) is klt. These conditions are not necessary, but they seem to apply in most cases of interest to us.

Lemma 22 (cf. [**KM98**, 4.5 and 5.50])**.** *Let S be a smooth surface and D an effective $\mathbb{Q}$-divisor. Then (S, D) is klt if the following conditions are satisfied.*

1) *D does not contain an irreducible component with coefficient ≥ 1.*

2) *For every point $P \in S$, either*
   a) $\operatorname{mult}_P D \le 1$, *or*
   b) *we can write $D = cC + D'$ where C is a curve through P, smooth at P, D' is effective not containing C, and the local intersection number $(C \cdot_P D') < 1$.*

4. Log Del Pezzo surfaces with large H^2

In this section we construct smooth log Del Pezzo surfaces with a Kähler–Einstein metric and $\operatorname{rank} H^2$ arbitrarily large. Methods to construct log Del Pezzo surfaces with large H^2 are given in [**KM99**, **McK03**]. For ordinary smooth Del Pezzo surfaces the rank of H^2 is at most 9.

Example 23. Start with $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinate projections π_1, π_2. Let $C_1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a fiber of π_2 and $C_2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ the graph of a degree 2 morphism $\mathbb{P}^1 \to \mathbb{P}^1$. Thus $\pi_1 : C_2 \to \mathbb{P}^1$ has degree 1 and $\pi_2 : C_2 \to \mathbb{P}^1$ has degree 2. Pick k points $P_1, \dots, P_k \in C_2 \setminus C_1$ and blow them up to obtain a surface $h : S_k \to \mathbb{P}^1 \times \mathbb{P}^1$. Let $C_i' \subset S_k$ denote the birational transform of C_i. Note that $(C_1')^2 = 0$, $(C_1' \cdot C_2') = 2$ and $(C_2')^2 = 4 - k$.
$C_1 + C_2 \in |-K_{\mathbb{P}^1 \times \mathbb{P}^1}|$, thus $C_1' + C_2' \in |-K_{S_k}|$.

Lemma 24. *Let a_i be rational numbers with $a_1 > \frac{k-4}{2} a_2 > 0$. Then a large multiple of $a_1 C_1' + a_2 C_2'$ determines a birational morphism $S_k \to \bar{S}_k$. The positive dimensional fibers are exactly the birational transforms of those fibers of π_2 which intersect C_2 in two points of the set $P_1, \dots, P_k$.*

Proof. The conditions ensure that $(a_1 C_1' + a_2 C_2') \cdot C_2' > 0$, thus $a_1 C_1' + a_2 C_2'$ is nef and big.
For $c > a_1, a_2$ we can write

$$a_1 C_1' + a_2 C_2' \equiv -c \left(K_{S_k} + (1 - a_1/c) C_1' + (1 - a_2/c) C_2' \right).$$

The Base point free theorem (cf. [**KM98**, 3.3]) applies and so a large multiple of $a_1 C_1' + a_2 C_2'$ determines a birational morphism.

The positive dimensional fibers are exactly those curves which have zero intersection number with $a_1 C_1' + a_2 C_2'$. The projection of such a curve to $\mathbb{P}^1 \times \mathbb{P}^1$ is thus a curve B which intersects $C_1 + C_2$ only at the points $P_1, \dots, P_k$. Since B is disjoint from C_1, it is the union of fibers of π_2. q.e.d.

Lemma 25. *Notation as above. Assume that $k \ge 5$ and choose natural numbers $m_1, m_2 \ge 2$ satisfying $m_2 > \frac{k-4}{2} m_1$. Assume that no two of the P_i are on the same fiber of π_2. Then*
1) *S_k is smooth, its Picard number is $k+2$.*
2) *C_1' and C_2' are smooth rational curves intersecting transversally and they form part of a basis of $H^2(S_k^*, \mathbb{Z})$.*
3) *$-(K_{S_k} + (1 - \frac{1}{m_1}) C_1' + (1 - \frac{1}{m_2}) C_2')$ is ample.*

4) $\pi_1(S_k \setminus (C_1' + C_2')) = 1$.

Proof. The canonical class of S_k is $-C_1' - C_2'$, thus

$$-(K_{S_k} + (1 - \tfrac{1}{m_1})C_1' + (1 - \tfrac{1}{m_2})C_2') = \tfrac{1}{m_1}C_1' + \tfrac{1}{m_2}C_2',$$

and this is ample by (24).

By looking at the first projection π_1 we see that $\mathbb{P}^1 \times \mathbb{P}^1 \setminus (C_1 + C_2)$ contains an open subset $\mathbb{C}^* \times \mathbb{C}^*$. Thus $\pi_1(S_k \setminus (C_1' + C_2'))$ is abelian and is generated by small loops around the C_i'.

Any of the exceptional curves of $S_k \to \mathbb{P}^1 \times \mathbb{P}^1$ intersects C_2' transversally in one point and is disjoint from C_1'. Similarly, the birational transform of $\mathbb{P}^1 \times \{P_i\}$ (for any i) intersects C_1' transversally in one point and is disjoint from C_2'. These show that C_1' and C_2' form part of a basis of $H^2(S_k, \mathbb{Z})$ and that the small loops around the C_i' are contractible in $S_k \setminus (C_1' + C_2')$, thus $\pi_1(S_k \setminus (C_1' + C_2')) = 1$. q.e.d.

The next result gives information about klt divisors on S_k.

Lemma 26. *Notation as above and assume that $k \geq 5$. Let $D \subset S_k$ be an effective $\mathbb{Q}$-divisor such that $[D] = [b_1C_1' + b_2C_2']$ for some $0 \leq b_i < 1/2$. Then $(S_k, \frac{1}{2}C_1' + \frac{1}{2}C_2' + D)$ is klt, except possibly at one, but not both, of the two intersection points $C_1' \cap C_2'$.*

Proof. Assume that $\frac{1}{2}C_1' + \frac{1}{2}C_2' + D$ is not klt at a point $Q \in S_k$. If Q is not on any of the exceptional curves of h then $h_*(\frac{1}{2}C_1' + \frac{1}{2}C_2' + D)$ is also not klt at $h(Q) \in \mathbb{P}^1 \times \mathbb{P}^1$. By assumption $[h_*D]$ is cohomologous to the sum of the two lines on $\mathbb{P}^1 \times \mathbb{P}^1$ with coefficients less than 1; these are always klt. Thus $\frac{1}{2}C_1' + \frac{1}{2}C_2' + D$ is klt outside $h^{-1}(C_1 + C_2)$.

Write

$$\tfrac{1}{2}C_1' + \tfrac{1}{2}C_2' + D = (\tfrac{1}{2} + d_1)C_1' + (\tfrac{1}{2} + d_2)C_2' + D',$$

where D' does not contain the C_i'. Then $D' \equiv (b_1 - d_1)C_1' + (b_2 - d_2)C_2'$ and $d_i \leq b_i$ since otherwise we would obtain that

$$[D'] = \pm[\alpha C_1' - \beta C_2'] \quad \text{with } \alpha, \beta > 0.$$

Both are impossible as they lead to a negative intersection number with one of the C_i'. In particular, $\frac{1}{2} + d_i < 1$, so the C_i' appear in $\frac{1}{2}C_1' + \frac{1}{2}C_2' + D$ with coefficient less than 1. Thus $\frac{1}{2}C_1' + \frac{1}{2}C_2' + D$ satisfies the condition (22.1) for C_1' and C_2'.

In order to check (22.2.b) along C_1' (resp. C_2'), we study when the restriction $(\frac{1}{2} + d_2)C_2' + D'|_{C_1'}$ (resp. $(\frac{1}{2} + d_1)C_1' + D'|_{C_2'}$) is klt. We compute the intersection numbers

$$\begin{aligned} \deg D'|_{C_1'} &= (D' \cdot C_1') = 2(b_2 - d_2) < 1, \quad \text{and} \\ \deg D'|_{C_2'} &= (D' \cdot C_2') = 2(b_1 - d_1) - (k-4)(b_2 - d_2) < 1. \end{aligned}$$

In both cases, the remainder of the restrictions consists of the 2 intersection points $Q_1 + Q_2 = C_1' \cap C_2'$, each with coefficient $\frac{1}{2} + d_2$ (resp.

$\frac{1}{2}+d_1)$. Therefore $\deg\left((\frac{1}{2}+d_{3-i})C'_{3-i}+D'|_{C'_i}\right)<2$ and the restriction contains both Q_1 and Q_2 with coefficient bigger than $\frac{1}{2}$.

Thus every point occurs with coefficient less than 1, except possibly for Q_1 and Q_2. Moreover, only one of these two points can have coefficient at least 1.

We are left to understand what happens along the exceptional curves of h. Let E be such a curve and write $D' = eE + D''$ where D'' does not contain E. Then

$$e \le (D' \cdot C'_2) \le 2(b_1-d_1)-(k-4)(b_2-d_2) < 1, \quad \text{and so}$$
$$(D'' \cdot E) = (D' \cdot E) + e \le 2(b_1-d_1)-(k-5)(b_2-d_2) < 1.$$

The first inequality shows (22.1) for E and the second shows that (22.2.b) holds at every point of E. q.e.d.

Remark 27. In the above proof we could have used the Connectedness theorem (cf. [**KM98**, 5.48]), which implies that the set of points where $(S_k, \frac{1}{2}C'_1+\frac{1}{2}C'_2+D)$ is not klt is connected.

This is, however, not enough to obtain a Kähler–Einstein metric on S_k. To achieve this, we make a special choice of the points P_i. It is easiest to write down everything by equations.

Definition 28. Choose homogeneous coordinates $\mathbb{P}^1_{(s:t)} \times \mathbb{P}^1_{(u:v)}$ and pick $C_1 = (u=v)$ and $C_2 = (s^2u = t^2v)$. The two intersection points Q_i are $(\pm 1:1, 1:1)$. The involution $\tau : (s:t, u:v) \mapsto (-t:s, v:u)$ fixes that C_i and interchanges the two points Q_i.

If $k = 2m$ is even, pick $0 < c_1 < \cdots < c_m < 1$ and for the P_i choose the $2m$ points

$$(c_i : 1, c_i^2 : 1) \quad \text{and} \quad (-1 : c_i, 1 : c_i^2),$$

to obtain a surface S_k^* of Picard number $k+2$.

If $k = 2m+1$ is odd, pick $0 < c_1 < \cdots < c_m < 1$ and for the P_i choose the $2m+1$ points

$$(c_i : 1, c_i^2 : 1), (-1 : c_i, 1 : c_i^2) \quad \text{and} \quad (\sqrt{-1} : 1, -1, 1),$$

to obtain a surface S_k^* of Picard number $k+2$.

In both cases, the involution τ lifts to an involution on S_k^*, again denoted by τ.

Lemma 29. *Notation as above and assume that $k \ge 5$. Let $D \subset S_k^*$ be an effective τ-invariant $\mathbb{Q}$-divisor such that $[D] = [b_1C'_1 + b_2C'_2]$ for some $0 \le b_i < 1/2$. Then $(S_k^*, \frac{1}{2}C'_1+\frac{1}{2}C'_2+D)$ is klt.*

Proof. We already know that $\frac{1}{2}C'_1+\frac{1}{2}C'_2+D$ is klt, except possibly at the two intersection points $C'_1 \cap C'_2$. These are interchanged by τ, so if $\frac{1}{2}C'_1+\frac{1}{2}C'_2+D$ is not klt, then it is not klt at exactly these 2 points. We have seen in (26) that it can not fail to be klt at both of these points. q.e.d.

Corollary 30. *Notation as above. Assume that $k \geq 5$ and choose natural numbers $m_1, m_2 \geq 2$ satisfying $m_2 > \frac{k-4}{2} m_1$. Then $(S_k^*, (1 - \frac{1}{m_1})C_1' + (1 - \frac{1}{m_2})C_2')$ has an orbifold Kähler–Einstein metric.*

Proof. Set $\Delta = (1 - \frac{1}{m_1})C_1' + (1 - \frac{1}{m_2})C_2'$. Then $-(K_{S_k^*} + \Delta) = \frac{1}{m_1}C_1' + \frac{1}{m_2}C_2'$ is ample by (24). For the existence of a Kähler–Einstein metric, we use the criterion (20) with $\frac{2}{3} + \epsilon = \frac{3}{4}$. Let D be any effective τ-invariant divisor numerically equivalent to $-(K_{S_k^*} + \Delta)$. Then

$$\Delta + \tfrac{3}{4}D \equiv \tfrac{1}{2}C_1' + \tfrac{1}{2}C_2' + (\tfrac{1}{2} - \tfrac{1}{m_1})C_1' + (\tfrac{1}{2} - \tfrac{1}{m_2})C_2' + \tfrac{3}{4}D.$$

Note that

$$(\tfrac{1}{2} - \tfrac{1}{m_1})C_1' + (\tfrac{1}{2} - \tfrac{1}{m_2})C_2' + \tfrac{3}{4}D \equiv (\tfrac{1}{2} - \tfrac{1}{4m_1})C_1' + (\tfrac{1}{2} - \tfrac{1}{4m_2})C_2'.$$

The assumptions of (29) are satisfied and so $(S_k^*, \Delta + \frac{3}{4}D)$ is klt. Thus (S, Δ) has an orbifold Kähler–Einstein metric. q.e.d.

5. Einstein metrics on $k\#(S^2 \times S^3)$

Let $C_1, C_2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ be as in (28). For $k \geq 6$ let M_{k-1} be the moduli space of $k-1$ distinct points in $C_2 \setminus C_1$. Its dimension is $k-1$. We can also think of M_{k-1} as parametrizing surfaces of type S_{k-1} obtained by blowing up these points.

Fix $k \geq 6$ and relatively prime odd numbers $m_1, m_2 > 2$ satisfying $m_2 > \frac{k-5}{2} m_1$. We can then further view M_{k-1} as parametrizing orbifolds

$$(S_{k-1}, (1 - \tfrac{1}{m_1})C_1' + (1 - \tfrac{1}{m_2})C_2').$$

Let $f : L \to S_{k-1}$ be the Seifert bundle with orbit invariants $(m_1, 1)$ along C_1', $(m_2, 1)$ along C_2' and with the trivial line bundle as B_L.

The number a in (5.6) is $m_1 m_2$. The Chern class of L is

$$c_1(L) = \tfrac{1}{m_1}[C_1'] + \tfrac{1}{m_2}[C_2'] = -\left(K_{S_{k-1}} + (1 - \tfrac{1}{m_1})C_1' + (1 - \tfrac{1}{m_2})C_2'\right).$$

Furthermore, $ac_1(L) = m_2[C_1'] + m_1[C_2']$ is not divisible since m_1, m_2 are relatively prime and computing modulo 2

$$ac_1(L) \equiv -[C_2'] - [C_1'] = [K_{S_{k-1}}] \equiv w_2(S)$$

since the m_i are odd.

Using (25) as well, we see that the conditions of (10) are all satisfied, thus any such L is diffeomorphic to $k\#(S^2 \times S^3)$.

The surfaces of type S_{k-1}^* considered in (30) form a subset of M_{k-1}, and for these surfaces we have proved the existence of an orbifold Kähler-Einstein metric. The existence of an orbifold Kähler–Einstein metric is an open condition (in the Euclidean topology), thus we obtain:

Claim 31. For $k \geq 6$ and odd $m_1, m_2 > 2$ satisfying $m_2 > \frac{k-5}{2} m_1$ there is an open subset $U(k-1, m_1, m_2) \subset M_{k-1}$ such that all orbifolds $(S_{k-1}, (1 - \frac{1}{m_1})C_1' + (1 - \frac{1}{m_2})C_2')$ corresponding to a point in $U(k-1, m_1, m_2)$ have an orbifold Kähler–Einstein metric.

By (19), for any surface corresponding to a point in $U(k-1, m_1, m_2)$ we obtain an Einstein metric on L. The space $U(k-1, m_1, m_2)$ has complex dimension $k-1$ hence real dimension $2k-2$.

Complement 32. As explained in [**BG00**] (see also [**BGK05**]), the Einstein metrics constructed this way have additional good properties.

1) The connected component of the isometry group of the metric is S^1.
2) All these metrics are *Sasaki–Einstein*.
3) Two metrics constructed from the data $(m_1^i, m_2^i, P_1^i, \dots, P_{k-1}^i)$ for $i = 1, 2$ are isometric iff $m_1^1 = m_1^2, m_2^1 = m_2^2$ and the point set $\{P_1^1, \dots, P_{k-1}^1\}$ can be mapped to either $\{P_1^2, \dots, P_{k-1}^2\}$ or to its conjugate by an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ fixing C_1 and C_2. (Such automorphisms form a group of order 4.)

References

[Bes87] A.L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **10**, Springer-Verlag, Berlin, 1987, MR 867684, Zbl 0613.53001.

[BG00] C.P. Boyer & K. Galicki, *On Sasakian-Einstein geometry*, Internat. J. Math. **11**(**7**) (2000) 873–909, MR 1792957, Zbl 1022.53038.

[BG03] ———, *New Einstein metrics on* $8\#(S^2 \times S^3)$, Differential Geom. Appl. **19**(**2**) (2003) 245–251, MR 2002662, Zbl 1042.53028.

[BGK05] C.P. Boyer, K. Galicki, & J. Kollár, *Einstein metrics on spheres*, Ann. of Math. **162** (2005) 1–24, MR 2178969.

[BGN02] C.P. Boyer, K. Galicki, & M. Nakamaye, *Sasakian-Einstein structures on* $9\#(S^2 \times S^3)$, Trans. Amer. Math. Soc. **354**(**8**) (2002) 2983–2996 (electronic), MR 1897386, Zbl 1003.53038.

[BGN03a] ———, *On positive Sasakian geometry*, Geom. Dedicata **101**(**1**) (2003) 93–102, MR 2017897, Zbl 1046.53029.

[BGN03b] ———, *On the geometry of Sasakian-Einstein* 5*-manifolds*, Math. Ann. **325**(**3**) (2003) 485–524, MR 1968604, Zbl 1046.53028.

[DK01] J.-P. Demailly & J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) **34**(**4**) (2001) 525–556, MR 1852009, Zbl 0994.32021.

[KOL05] J. Kollár, *Einstein metrics on five-dimensional Seifert bundles*, J. Geom. Anal. **15**(**3**) (2005) 445–476, MR 2190241.

[KM98] J. Kollár & S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, **134**, Cambridge University Press, Cambridge, 1998, with the collaboration of C.H. Clemens and A. Corti, translated from the 1998 Japanese original, MR 1658959, Zbl 0926.14003.

[KM99] S. Keel & J. McKernan, *Rational curves on quasi-projective surfaces*, Mem. Amer. Math. Soc. **140**(**669**) (1999), MR 1610249, Zbl 0955.14031.

[Kob63] S. Kobayashi, *Topology of positively pinched Kaehler manifolds*, Tôhoku Math. J. (2) **15** (1963) 121–139, MR 0154235, Zbl 0114.37601.

[McK03] J. McKernan, *A density property of log Del Pezzos*, manuscript, 2003.

[MS74] J.W. Milnor & J.D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, NJ, 1974, MR 0440554, Zbl 0298.57008.

[Nad90] A.M. Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math. (2) **132**(**3**) (1990) 549–596, MR 1078269, Zbl 0731.53063.

[OW75] P. Orlik & P. Wagreich, *Seifert n-manifolds*, Invent. Math. **28** (1975) 137–159, MR 0361150, Zbl 0295.57021.

[Sma62] S. Smale, *On the structure of* 5*-manifolds*, Ann. of Math. (2) **75** (1962) 38–46, MR 0141133, Zbl 0101.16103.

[SY91] J.-P. Sha & DaGang Yang, *Positive Ricci curvature on the connected sums of* $S^n \times S^m$, J. Differential Geom. **33**(**1**) (1991) 127–137, MR 1085137, Zbl 0728.53027.

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